Using Differentials to Study Population Dynamics

We have seen that differentials give a convenient way for expressing linear approximations. In this example, we explore population dynamics in the language of differentials.

A simple generational model of population dynamics says that an initial population x will yield a next generation with population given by a function $P(x)$. The next generation after that is given by "iterating" the function P, that is, $P(P(x))$. We can keep applying P to the result to find the population of successive generations. Note in particular that population will be stable over generations at any x such that $P(x) = x$. Such an x is known as a "fixed point."

We say that a fixed point x_0 is "attracting" if, given an initial population value $x_0 + \Delta x$ with Δx sufficiently small, the successive generations have size closer and closer to x_0. More formally, the sequence of values

$$x_0 + \Delta x, P(x_0 + \Delta x), P(P(x_0 + \Delta x)), P(P(P(x_0 + \Delta x))), \ldots$$

gets closer and closer to x_0.

Question:

- Show that if x_0 is a fixed point of $P(x)$ and $|P'(x_0)| < 1$, then x_0 is attracting.
- Given fixed positive constants a, b with $ab > 1$, find the fixed points of $P(x) = ax(b - x)$ and determine if they are attracting.

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If x_0 is a fixed point of $P(x)$,

then $P(x_0) = x_0$.

$dy = P'(x_0)\,dx$

$P(x_0+\Delta x) \approx P(x_0 + dx)$

$= y + dy$

$= x_0 + P'(x_0)\,dx$

$|P'(x_0)| < 1 \Rightarrow |P'(x_0)\,dx| < 1 \Rightarrow |x_0 - P(x_0+\Delta x)| < |x_0 - (x_0+\Delta x)|$

Define $P'(x_0+\Delta x) \approx x_0 + P'(x_0)\,dx_1$,

then $P^n(x_0+\Delta x) \approx x_0 + P^n(x_0)\,dx_n$.

$\Rightarrow$ As $n \to \infty$, $P^n(x_0) \to 0 \Rightarrow P^n(x_0+\Delta x) \approx x_0$ as $n \to \infty$. $\square$

$a, b \; ; \; ab > 1$

$p(x) = ax(b-x)$

Fixed points $x \Rightarrow p(x) = x$

$\Rightarrow x = ax(b-x)$

$abx - ax^2 - x = 0$

$-ax^2 + (ab-1)x = 0$

$(-ax + (ab-1))x = 0$

$x = 0, \ \frac{ab-1}{a}$

$p'(x) = a(b-x) + ax(-1)$

$= ab - ax - ax$

$= ab - 2ax$

$= a(b-2x)$

$p'(b-\frac{1}{a}) = a(b - 2(b-\frac{1}{a}))$

$= ab - 2ab + 2$

$= -ab + 2$

$p'(0) = ab$

$\Rightarrow ab > 1$

$p'(0) > 1$

$\left| p'(b-\frac{1}{a}) \right| < 1$

$|2-ab| < 1$

$\Rightarrow 2 - ab < 1$

$ab > 2 - 1$

$ab > 1$

$-(2-ab) < 1$

$2 - ab > -1$

$ab < 2 + 1$

$ab < 3$

$\therefore$ Fixed point $x=0$ is attractive for any $ab>1$,

$x = b - \frac{1}{a}$ is attractive for $1 < ab < 3$.